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An entire function and its derivatives sharing a polynomial ☆

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Abstract

In this paper, we study the growth of solutions of a first-order linear differential equation and that of a second-order linear differential equation. From this we obtain some uniqueness theorems of a nonconstant entire function and its first derivative having the same fixed points with the same multiplicities. The results in this paper also improve some known results. Some examples show that the results in this paper are best possible.

Keywords: Entire function; Order of growth; Shared value; Uniqueness

1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1–3]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o\big(T(r,h)\big) \quad (r \to \infty,\ r \notin E).$$

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Let f and g be two nonconstant meromorphic functions, and let P be a polynomial or a finite complex number. We say that f and g share the value P CM, provided that $f - P$ and $g - P$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value P IM, provided that $f - P$ and $g - P$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share the value 0 CM, and we say that f and g share ∞ IM, if f and g share the value 0 IM (see [4]). In this paper, we also need the following two definitions.

Definition 1.1. Let f be a nonconstant meromorphic function, the order of f, denoted $\sigma(f)$, is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Remark 1.1. Clearly, if f is an entire function, then

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r}\{|f(z)|\}$.

Definition 1.2. Let f be a nonconstant meromorphic function, the hyper-order of f, denoted $\nu(f)$, is defined by

$$\nu(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

In 1977, L. Rubel and C.C. Yang proved the following theorem.

Theorem A. (See [5].) *Let f be a nonconstant entire function. If f and f' share two finite distinct values CM, then $f \equiv f'$.*

In 1996, R. Brück proved the following theorems.

Theorem B. (See [6].) *Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer. If f and f' share the value 0 CM, then $f \equiv cf'$ for some constant $c \neq 0$.*

Theorem C. (See [6].) *Let f be a nonconstant entire function. If f and f' share the value 1 CM, and if $N(r, 1/f') = S(r, f)$, then $f - 1 \equiv c(f' - 1)$ for some constant $c \neq 0$.*

Brück made the following conjecture.

Conjecture 1.1. (See [6].) *Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, where $\nu(f)$ is not a positive integer. If f and f' share one finite value a CM, then $f - a \equiv c(f' - a)$ for some constant $c \neq 0$.*

Consider the differential equation

$$F' - e^{Q(z)}F = 1, \tag{1.1}$$

where $Q(z)$ is an entire function.

In 1998, G.G. Gundersen and Lian-Zhong Yang proved that the conjecture is true for $a \neq 0$, provided that f satisfies the additional assumption $\sigma(f) < \infty$. In fact, they proved the following results.

Theorem D. (See [7, Lemma 1].) *Let $Q(z)$ be a nonconstant polynomial. Then every solution of* (1.1) *is an entire function of infinite order.*

Theorem E. (See [7, Theorem 1].) *Let f be a nonconstant entire function of finite order. If f and f' share one finite value a CM, then $f - a \equiv c(f' - a)$ for some constant $c \neq 0$.*

In this paper, we shall prove the following results, which are some improvements and supplements of Theorems D and E.

Theorem 1.1. *Let $Q_j(z)$ $(j = 1, 2)$ be a polynomial with degree $\gamma_{Q_j} \geqslant 1$ $(j = 1, 2)$, and let $P(z)$ be a polynomial. If f is a nonconstant solution of the equation*

$$\frac{f' - Q_1}{f - Q_2} = e^{P(z)}, \tag{1.2}$$

then $\nu(f) = \gamma_P$, where γ_P is the degree of $P(z)$.

From Theorem 1.1 we can get the following one result on the growth of a nonconstant solution of a second-order linear differential equation.

Corollary 1.1. *Let $Q(z)$ be a polynomial with degree $\gamma_Q \geqslant 1$, and let $P(z)$ be a polynomial. If f is a nonconstant solution of the differential equation*

$$f'' - e^P \cdot f' - P'e^P \cdot f + (QP' + Q') \cdot e^P - Q' = 0, \tag{1.3}$$

then $\nu(f) = \gamma_P$, where γ_P is the degree of $P(z)$.

Proof. Since f is a solution of (1.3), thus from (1.3) we can see that there exists some finite complex constant c such that

$$\frac{f' - (Q + c)}{f - Q} = e^P. \tag{1.4}$$

From (1.4) and Theorem 1.1 we can get the conclusion of Corollary 1.1. $\square$

From Theorem 1.1 we also get the following two corollaries.

Corollary 1.2. *Let $Q_j(z)$ $(j = 1, 2)$ be a polynomial with degree $\gamma_{Q_j} \geqslant 1$ $(j = 1, 2)$, and let $P(z)$ be a polynomial, if f is a solution of* (1.2) *such that $\sigma(f) = \infty$, then $P(z)$ is a nonconstant polynomial and $\nu(f) = \gamma_P$, where γ_P is the degree of $P(z)$.*

Corollary 1.3. *Let $Q(z)$ be a polynomial with degree $\gamma_Q \geqslant 1$, and let P be a polynomial. If f is a nonconstant solution of the differential equation*

$$\frac{f' - Q}{f - Q} = e^P, \tag{1.5}$$

such that $\nu(f)$ is not a positive integer, then $f - Q \equiv c(f' - Q)$ for some constant $c \neq 0$.

Proof. First, from (1.5) and Theorem 1.1 we can get

$$\nu(f) = \gamma_P. \tag{1.6}$$

On the other hand, since $\nu(f)$ is not a positive integer, thus from (1.6) we can get the conclusion of Corollary 1.3. □

From Corollary 1.3 we can get the following one result.

Corollary 1.4. *Let f be a nonconstant entire function of finite order, and let $Q(z)$ be a polynomial with degree $\gamma_Q \geqslant 1$, if f and f' share Q CM, then $f - Q \equiv c(f' - Q)$ for some constant $c \neq 0$.*

Now we give the following one example.

Example 1.1. Let f be a solution of the differential equation

$$\frac{f'(z) - z}{f(z) - z} = e^{z^n},$$

where n is a positive integer. Then we can see that f is a nonconstant entire function, and that $f(z) - z$ and $f'(z) - z$ share the value 0 CM. Moreover, Theorem 1.1 immediately yields $\nu(f) = \sigma(e^{z^n}) = n$. This example shows that the conclusion of Theorem 1.1 and that of Corollary 1.2 can occur. This example also shows that the condition "$\nu(f)$ is not a positive integer" in Corollary 1.3 is best possible.

From Corollary 1.4 we can get the following one result.

Corollary 1.5. *Let f be a nonconstant entire function of finite order. If f and f' have the same fixed points with the same multiplicities, then $f(z) - z \equiv c(f'(z) - z)$ for some constant $c \neq 0$.*

From Corollary 1.3 we also get the following four corollaries, which are some supplements of the results in the paper of L.Z. Yang [8].

Corollary 1.6. *Let $Q(z) = z$, and let $P(z)$ be a polynomial. If f is a nonconstant solution of the differential equation* (1.5) *such that $\nu(f)$ is not a positive integer, and if there exists one point z_0 such that $f'(z_0) = f(z_0) \neq z_0$, then $f \equiv f'$.*

Corollary 1.7. *Let $Q(z) = z$, and let $P(z)$ be a polynomial, and let a $(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation* (1.5) *such that $\nu(f)$ is not a positive integer, and if f and f' share the value a IM, then $f \equiv f'$.*

Proof. Since f and f' share the value a IM, thus by Hayman's inequality (see [1, Theorem 3.5]) we can see that there exists one point z_0 such that $f'(z_0) = f(z_0) = a \neq z_0$. Thus from Corollary 1.6 we can get the conclusion of Corollary 1.7. □

Corollary 1.8. *Let $Q(z) = z$, and let $P(z)$ be a polynomial. If f is a nonconstant solution of the differential equation* (1.5) *such that $\nu(f)$ is not a positive integer, and if f and f' share the value* 0 *IM, then $f \equiv f'$.*

Proof. First, from Corollary 1.3 we can get

$$\frac{f'(z)-z}{f(z)-z} \equiv c, \tag{1.7}$$

where c is some finite nonzero constant. We discuss the following two cases.

Case 1. Suppose that f is a transcendental entire function. From (1.7) we can get

$$f^{(3)}(z) - cf''(z) \equiv 0. \tag{1.8}$$

From (1.8) we can deduce

$$f' = A_1 e^{cz} + A_2, \tag{1.9}$$

where A_1 ($\neq 0$) and A_2 are two finite complex constants. From (1.9) we can deduce

$$f = \frac{A_1}{c} \cdot e^{cz} + A_2 z + A_3, \tag{1.10}$$

where A_3 is a finite complex constant. We discuss the following two subcases.

Subcase 1.1. Suppose that $A_2 \neq 0$. Since f and f' share the value 0 IM, thus from (1.9) and (1.10) we can deduce

$$\begin{aligned} N\left(r, \frac{1}{A_1 e^{cz} + A_2}\right) &= N\left(r, \frac{1}{f'}\right) \leqslant N\left(r, \frac{1}{cf - f'}\right) \\ &= N\left(r, \frac{1}{A_2 cz + A_3 c - A_2}\right) \leqslant O(\log r), \end{aligned}$$

this is impossible.

Subcase 1.2. Suppose that $A_2 = 0$. Since f and f' share the value 0 IM, thus from (1.7), (1.9) and (1.10) we can get

$$\frac{A_1 e^{cz} - z}{A_1 e^{cz} - cz} \equiv 1,$$

which implies that $c = 1$, so from (1.7) we can get $f \equiv f'$.

Case 2. Suppose that f is a polynomial. Then from (1.7) we can see that

$$f = B_1 z + B_2, \tag{1.11}$$

where B_1 ($\neq 0$) and B_2 are two finite complex constants. From (1.7) and (1.11) we can deduce

$$c \neq 1, \tag{1.12}$$

$$B_1 = 1 - \frac{1}{c} \tag{1.13}$$

and

$$B_2 = \left(1 - \frac{1}{c}\right) \cdot \frac{1}{c}. \tag{1.14}$$

From (1.11)–(1.14) we easily get

$$f = \left(1 - \frac{1}{c}\right) z + \left(1 - \frac{1}{c}\right) \cdot \frac{1}{c},$$

where c ($\neq 0, 1$) is a finite complex constant. From this we easily see that f and f' do not share the value 0, this contradicts the assumption of Corollary 1.8.

Corollary 1.8 is thus completely proved. □

Corollary 1.9. *Let $Q(z) = z$, and let $P(z)$ be a polynomial. If f is a nonconstant solution of the differential equation* (1.5) *such that $\nu(f)$ is not a positive integer, and if there exists one point z_0 such that $f'(z_0) = f''(z_0) \neq 1$, then $f \equiv f'$.*

Corollary 1.10. *Let $Q(z) = z$, and let $P(z)$ be a polynomial, and let n ($\geqslant 2$) be a positive integer. If f is a nonconstant solution of the differential equation* (1.5) *such that $\nu(f)$ is not a positive integer, and if there exists one point z_0 such that $f^{(n)}(z_0) = f^{(n+1)}(z_0) \neq 0$, then $f \equiv f'$.*

In 1995, H.X. Yi and C.C. Yang posed the following question named question of Yi and Yang.

Question 1.1. (See [4, p. 458].) *Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex constant. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m ($n < m$) are distinct positive integers not all even or odd, then can we get the result $f \equiv f^{(n)}$?*

Regarding Question 1.1, G.G. Gundersen and Lian-Zhong Yang proved the following result in 1998.

Theorem F. (See [7, Theorem 2].) *Let f be a nonconstant entire function of finite order, let a ($\neq 0$) be a finite constant, and let n be a positive integer. If the value a is shared by f, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f \equiv f'$.*

In this paper, we shall prove the following one result, which is an improvement and supplement of Theorem F.

Theorem 1.2. *Let $P(z)$ be a polynomial, and let n be a positive integer. If f is a nonconstant solution of the differential equation*

$$\frac{f^{(n+1)}(z) - z}{f^{(n)}(z) - z} = e^P, \tag{1.15}$$

such that $\nu(f)$ is not a positive integer, and if $f(z) - z$ and $f^{(n)}(z) - z$ share the value 0 *IM, then $e^P \equiv 1$, and f is given as one of the following two expressions*:

(i) *$f = z + \gamma_1 z^k$, where γ_1 ($\neq 0$) is a certain finite complex constant, and k satisfying $1 \leqslant k \leqslant n - 1$ is a positive integer.*
(ii) *$f = \gamma_2 e^z$, where γ_2 ($\neq 0$) is a certain finite complex constant.*

2. Some lemmas

Lemma 2.1. (See [2, Theorem 3.1] or [9, pp. 36–37].) *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r},$$

where $\nu(r, f)$ denotes the central-index of $f(z)$.

Lemma 2.2. (See [10, Lemma 2] or [11, Lemma 4].) *If f is a transcendental entire function of hyper-order $\nu(f)$, then*

$$\nu(f) = \limsup_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r},$$

where $\nu(r, f)$ denotes the central-index of $f(z)$.

Lemma 2.3. (See [12] or [4, Corollary of Theorem 1.20].) *Suppose that $f(z)$ is meromorphic in the complex plane. Then*

$$T(r, f) \leqslant O\big(T(2r, f') + \log r\big),$$

as $r \to \infty$.

Lemma 2.4. (See [2, Lemma 1.1.1].) *Let $g:(0,+\infty) \to R$, $h:(0,+\infty) \to R$ be monotone increasing functions such that $g(r) \leqslant h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leqslant h(\alpha r)$ for all $r > r_0$.*

Lemma 2.5. (See [13, Lemma 4].) *Let $f_1, f_2, \ldots, f_n$ be nonconstant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r), \quad i = 1, 2, \ldots, n,$$

and

$$T(r, f_i) \neq S(r), \qquad T\left(r, \frac{f_i}{f_j}\right) \neq S(r), \quad i \neq j,\ i, j = 1, 2, \ldots, n.$$

Let $a_0, a_1, \ldots, a_m$ $(m \leqslant n)$ be meromorphic functions satisfying $T(r, a_i) = S(r)$, $i = 0, 1, \ldots, m$. If

$$\sum_{i=1}^{m} a_i f_i \equiv a_0,$$

then $a_0 \equiv a_1 \equiv \cdots \equiv a_m \equiv 0$, where $S(r) = o(T(r))$, as $r \to \infty$ and $r \notin E$, and $T(r) = \sum_{i=1}^{n} T(r, f_i)$.

3. Proof of theorems

Proof of Theorem 1.1. Suppose that f is a polynomial, then from (1.2) we can see that there exists a certain nonzero constant c such that $e^{P(z)} \equiv c$. So $\nu(f) = \gamma_P = 0$, thus the conclusion of Theorem 1.1 is valid. Next we suppose that f is a transcendental entire function. We discuss the following two cases.

Case 1. Suppose that

$$\sigma(f) = \infty. \tag{3.1}$$

From (3.1) and Lemma 2.1 we can see that

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r} = \infty, \tag{3.2}$$

where $\nu(r,f)$ denotes the central-index of $f(z)$. Noting that $P(z)$ is a polynomial, we let

$$P(z)=p_nz^n+p_{n-1}z^{n-1}+\cdots+p_1z+p_0, \tag{3.3}$$

where $p_n, p_{n-1},\ldots,p_1$ and p_0 are finite complex constants. From (1.2) and (3.1) we can deduce $P(z)$ is a nonconstant polynomial. In fact, if $P(z)$ is a constant, then by Theorem 4.1 in the book of Laine [2], we easily deduce that the solutions of $f'-e^Pf=Q_1-e^PQ_2$ have finite order. Thus $p_n\neq 0$. Again from (3.3) we can easily get

$$\lim_{|z|\to+\infty}\frac{|P(z)|}{|p_nz^n|}=1. \tag{3.4}$$

From (3.4) we can easily see that there exists some sufficiently large positive number r_0, such that

$$\frac{|P(z)|}{|p_nz^n|}>\frac{1}{e}\quad\bigl(|z|>r_0\bigr). \tag{3.5}$$

From (1.2) and (3.5) we can easily deduce

$$\begin{aligned} n\log r+\log|p_n|-1=\log\frac{|p_nz^n|}{e}&\leqslant\log\bigl|P(z)\bigr|=\log\bigl|\log e^{P(z)}\bigr|\\ &\leqslant\bigl|\log\log e^{P(z)}\bigr|=\left|\log\log\frac{f'-Q_1}{f-Q_2}\right|\quad\bigl(|z|>r_0\bigr). \end{aligned} \tag{3.6}$$

On the other hand, from (3.1) we can see that f is a transcendental entire function. Thus

$$M(r,f)\to+\infty, \tag{3.7}$$

as $r\to+\infty$, where $M(r,f)=\max_{|z|=r}|f(z)|$. Again let

$$M(r,f)=\bigl|f(z_r)\bigr|, \tag{3.8}$$

where $z_r=re^{i\theta(r)}$, and $\theta(r)\in[0,2\pi)$ is some nonnegative real number. From (3.8) and the Wiman–Valiron theory (see [2, Theorem 3.2]), we can see that there exists a subset $E_1\subset(1,\infty)$ with finite logarithmic measure, i.e., $\int_{E_1}\frac{dt}{t}<\infty$, such that for some point $z_r=re^{i\theta(r)}$ ($\theta(r)\in[0,2\pi)$) satisfying $|z_r|=r\notin E_1$ and $M(r,f)=|f(z_r)|$, we have

$$\frac{f'(z_r)}{f(z_r)}=\frac{\nu(r,f)}{z_r}\bigl(1+\circ(1)\bigr), \tag{3.9}$$

as $r\to+\infty$, where $\nu(r,f)$ denotes the central-index of $f(z)$. Since f is a transcendental entire function, and Q_j $(j=1,2)$ is a polynomial with degree $\gamma_{Q_j}\geqslant 1$ $(j=1,2)$, thus from (3.1) and (3.8) we can deduce

$$\lim_{r\to\infty}\frac{|Q_j(z_r)|}{|f(z_r)|}=\lim_{r\to\infty}\frac{|Q_j(z_r)|}{M(r,f)}=0\quad(j=1,2). \tag{3.10}$$

Since

$$\frac{f'-Q_1}{f-Q_2}=\frac{\frac{f'}{f}-\frac{Q_1}{f}}{1-\frac{Q_2}{f}}, \tag{3.11}$$

thus from (3.2), (3.6)–(3.11) we can easily deduce

$$n\log|z_r|+\log|p_n|-1\leqslant\left|\log\log\left(\left(\frac{\nu(r,f)}{z_r}\right)\bigl(1+\circ(1)\bigr)\right)\right| \tag{3.12}$$

and

$$\begin{aligned}
&\log\left(\left(\frac{\nu(r,f)}{z_r}\right)\bigl(1+\circ(1)\bigr)\right)\\
&\quad=\log\nu(r,f)-\log re^{i\theta(r)}+\circ(1)\\
&\quad=\log\nu(r,f)-\log r-i\theta(r)+\circ(1)\\
&\quad=\left(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)}\right)\log\nu(r,f)+\circ(1),
\end{aligned}\tag{3.13}$$

as $r\to+\infty$. Thus, noting that $\theta(r)\in[0,2\pi)$, from (3.2), (3.13) and Lemma 2.2 we can easily deduce

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\left|\log\log\bigl(\bigl(\frac{\nu(r,f)}{z_r}\bigr)(1+\circ(1))\bigr)\right|}{\log r}\\
&\quad\leqslant\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}+\limsup_{r\to\infty}\frac{\left|\log\bigl(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)}\bigr)\right|}{\log r}\\
&\qquad+\lim_{r\to\infty}\frac{\log 2}{\log r}+\lim_{r\to\infty}\frac{2k_1\pi}{\log r}\\
&\quad=\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\nu(f),
\end{aligned}\tag{3.14}$$

where k_1 is some nonnegative integer. Noting that $|z_r|=r$, from (3.12) and (3.14) we can easily deduce

$$n\leqslant\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\nu(f).\tag{3.15}$$

Since $P(z)$ is a polynomial satisfying (3.3), thus

$$\sigma\bigl(e^{P(z)}\bigr)=\gamma_{P(z)}=n.\tag{3.16}$$

From (3.15) and (3.16) we can get

$$\sigma\bigl(e^{P(z)}\bigr)\leqslant\nu(f).\tag{3.17}$$

On the other hand, from (1.2), (3.9)–(3.11) we can deduce

$$\frac{\nu(r,f)}{z_r}\bigl(1+\circ(1)\bigr)=e^{P(z_r)},\tag{3.18}$$

as $r\to\infty$. So from (3.2) and (3.18) we can easily deduce

$$\begin{aligned}
\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}&=\limsup_{r\to\infty}\frac{\log\log\frac{\nu(r,f)}{2r}}{\log r}\\
&\leqslant\limsup_{r\to\infty}\frac{\log\log\bigl(\frac{\nu(r,f)}{|z_r|}\cdot|1+\circ(1)|\bigr)}{\log r}\\
&\leqslant\limsup_{r\to\infty}\frac{\log\log M(r,e^{P(z)})}{\log r}.
\end{aligned}\tag{3.19}$$

From (3.19), Lemma 2.2 and the definition of the order of an entire function we can get

$$\nu(f)\leqslant\sigma\bigl(e^{P}\bigr).\tag{3.20}$$

From (3.16), (3.17) and (3.20) we can get the conclusion of Theorem 1.1.

Case 2. Suppose that

$$\sigma(f) < \infty. \tag{3.21}$$

First, from (3.21) we can deduce

$$\nu(f) = 0. \tag{3.22}$$

On the other hand, from (1.2), (3.7)–(3.11), (3.21) and Lemma 2.1 we can deduce

$$\begin{aligned} \left|P(z_r)\right| = \left|\log e^{P(z_r)}\right| &= \left|\log \nu(r, f) - \log r e^{i\theta(r)} + \circ(1)\right| \\ &= \left|\log \nu(r, f) - \log r - i\theta(r) + \circ(1)\right| \\ &\leqslant O(\log r), \end{aligned} \tag{3.23}$$

as $r \to +\infty$. Since $P(z)$ is a polynomial, thus from (3.23) we can deduce $P(z)$ is a constant, and so $\gamma_P = 0$. From this and (3.22) we can see that the conclusion of Theorem 1.1 is valid.

Theorem 1.1 is thus completely proved. □

Proof of Theorem 1.2. Suppose that f is a nonconstant polynomial, then it follows by (1.15) that f is a polynomial of degree $\leqslant n-1$ or degree $n+1$. Assume that $\deg(f) = n+1$, where $\deg(f)$ denotes the degree of f, then it is deduced by (1.15) that

$$f^{(n)}(z) = az + a(1-a), \tag{3.24}$$

where a ($\neq 0, 1$) is a finite complex constant. On the other hand, by (3.24) and the assumption that $f(z) - z$ and $f^{(n)}(z) - z$ share the value 0 IM, we easily deduce

$$f(z) = z + \frac{a}{(n+1)!} \cdot (z-a)^{n+1}. \tag{3.25}$$

If $n = 1$, from (3.25) we easily deduce $f'(z) = 1 + a(z-a)$, which contradicts (3.24). If $n \geqslant 2$, from (3.25) we easily deduce $f^{(n)}(z) = a(z-a)$, which also contradicts (3.24). Thus f is a nonconstant polynomial of degree $\leqslant n-1$, so $f^{(n)}(z) = 0$. Again from the assumption that $f(z) - z$ and $f^{(n)}(z) - z$ share the value 0 IM, we easily deduce the conclusion (i) of Theorem 1.2. Next we suppose that f is a transcendental entire function, and so $f^{(k)}$ is also a transcendental entire function, where k is an arbitrary positive integer. From Lemma 2.3 we have

$$T(r, f) \leqslant O\big(T(2r, f') + \log r\big), \tag{3.26}$$

as $r \to \infty$. Noting that f and f' are transcendental entire functions, from (3.26) and the definition of the hyper-order of a nonconstant entire function we easily deduce

$$\nu(f) \leqslant \nu(f'). \tag{3.27}$$

On the other hand, since

$$T(r, f') \leqslant 2T(r, f) + O\big(\log r T(r, f)\big) \quad (r \notin E), \tag{3.28}$$

from (3.28) and Lemma 2.4 we easily deduce

$$\nu(f') \leqslant \nu(f). \tag{3.29}$$

From (3.27) and (3.29) we get

$$\nu(f) = \nu(f'). \tag{3.30}$$

Similarly

$$\nu\big(f^{(j)}\big) = \nu\big(f^{(j+1)}\big) \quad (1 \leqslant j \leqslant n-1). \tag{3.31}$$

From (3.30) and (3.31) we get

$$\nu(f) = \nu\big(f^{(n)}\big). \tag{3.32}$$

Since $\nu(f)$ is not a positive integer, from (1.15), (3.32) and Corollary 1.3 we can easily see that there exists a finite nonzero complex constant d, such that

$$\frac{f^{(n+1)}(z) - z}{f^{(n)}(z) - z} = e^P \equiv d. \tag{3.33}$$

Let

$$f^{(n)}(z) = g(z). \tag{3.34}$$

From (3.33) and (3.34) we can get

$$\frac{g' - z}{g - z} = d. \tag{3.35}$$

From (3.35) we can easily deduce

$$g^{(3)} - dg'' = 0. \tag{3.36}$$

From (3.36) we can get the characteristic equation

$$\lambda^3 - d\lambda^2 = 0. \tag{3.37}$$

Since the general solution of (3.36) has the form

$$f^{(n)}(z) = g(z) = \gamma_2 e^{dz} + b_{n+1} z + b_n, \tag{3.38}$$

with a certain nonzero constant γ_2, where d is the nonzero solution of (3.37), and b_{n+1} and b_n are finite complex constants, thus

$$f(z) = \frac{\gamma_2 e^{dz}}{d^n} + \frac{b_{n+1} z^{n+1}}{(n+1)!} + \frac{b_n z^n}{n!} + \sum_{j=0}^{n-1} b_j z^j, \tag{3.39}$$

where $b_0, b_1, b_2, \ldots, b_{n-1}$ are finite complex constants. Suppose that

$$d^n f(z) - f^{(n)}(z) \equiv 0. \tag{3.40}$$

Substituting (3.38) and (3.39) into (3.40) we easily deduce

$$b_j = 0 \quad (0 \leqslant j \leqslant n+1). \tag{3.41}$$

Substituting (3.41) into (3.39) we get

$$f(z) = \frac{\gamma_2 e^{dz}}{d^n},$$

which reveals that f has the desired form in the conclusion (ii) of Theorem 1.2. Next we suppose that

$$d^n f(z) - f^{(n)}(z) \not\equiv 0. \tag{3.42}$$

From (3.38), (3.39) and (3.42) we easily see that $d^n f(z) - f^{(n)}(z)$ is a nonzero polynomial. Noting that $f(z) - z$ and $f^{(n)}(z) - z$ share the value 0, from (3.38) and (3.39) we easily deduce that a common zero of $f(z) - z$ and $f^{(n)}(z) - z$ is a zero of the polynomial $d^n f(z) - f^{(n)}(z)$, so there can exist only finitely many such common zeros. Thus there exists a nonzero rational function $R(z)$, and a finite complex constant A such that

$$\frac{f(z) - z}{f^{(n)}(z) - z} = R(z)e^{Az}. \tag{3.43}$$

Substituting (3.38) and (3.39) into (3.43) we can get

$$\begin{aligned} &\frac{\gamma_2 e^{dz}}{d^n} - \gamma_2 R(z)e^{(d+A)z} + (z - b_{n+1}z - b_n)R(z)e^{Az} \\ &\quad = z - \frac{b_{n+1}z^{n+1}}{(n+1)!} - \frac{b_n z^n}{n!} - \sum_{j=0}^{n-1} b_j z^j. \end{aligned} \tag{3.44}$$

We discuss in the following two cases.

Case 1. Suppose that $A = d$. Then from (3.44) we can deduce

$$T\big(r, e^{dz}\big) = O(\log r), \tag{3.45}$$

which is impossible.

Case 2. Suppose that $A \neq d$. We discuss the following four subcases.

Subcase 2.1. Suppose that $A = -d$. Then (3.44) can be rewritten as

$$\begin{aligned} &\frac{\gamma_2 e^{2dz}}{d^n} + \left(\frac{b_{n+1}z^{n+1}}{(n+1)!} + \frac{b_n z^n}{n!} + \sum_{j=0}^{n-1} b_j z^j - z - \gamma_2 R(z)\right)e^{dz} \\ &\quad + (z - b_{n+1}z - b_n)R(z) = 0, \end{aligned}$$

from which we can get (3.45). This is impossible.

Subcase 2.2. Suppose that $A \neq -d$ and $A \neq 0$. Then from (3.44) and Lemma 2.5 we can get $\gamma_2 = 0$, this is impossible.

Subcase 2.3. Suppose that $n = 1$ and $A = 0$. Then (3.44) can be rewritten as

$$\left(\frac{\gamma_2}{d} - \gamma_2 R(z)\right)e^{dz} = -\frac{b_2 z^2}{2} + (1 - b_1)z - b_0 + (b_2 z - z + b_1)R(z). \tag{3.46}$$

By an elementary comparison of the growth of the left-hand and the right-hand side of (3.46), we can deduce $R(z) = \frac{1}{d}$,

$$b_2 = 0, \tag{3.47}$$

$$b_1 = 1 - \frac{1}{d} \tag{3.48}$$

and

$$b_0 = \frac{1}{d} \cdot \left(1 - \frac{1}{d}\right). \tag{3.49}$$

Substituting (3.47)–(3.49) into (3.39) we can get

$$f(z) = \frac{\gamma_2 e^{dz}}{d} + \left(1 - \frac{1}{d}\right)z + \frac{1}{d} \cdot \left(1 - \frac{1}{d}\right). \tag{3.50}$$

Thus from (3.33), (3.50) and $n=1$ we can deduce

$$\frac{f''(z)-z}{f'(z)-z}=\frac{\gamma_2 de^{dz}-z}{\gamma_2 e^{dz}+1-\frac{1}{d}-z}\equiv d,$$

which implies that $d=1$. From this and (3.50) we can get the conclusion (ii) of Theorem 1.2.

Subcase 2.4. Suppose that $n\geqslant 2$ and $A=0$. Then (3.44) can be rewritten as

$$\left(\frac{\gamma_2}{d^n}-\gamma_2 R(z)\right)e^{dz}=z-\frac{b_{n+1}z^{n+1}}{(n+1)!}-\frac{b_n z^n}{n!}-\sum_{j=0}^{n-1}b_j z^j+(b_{n+1}z-z+b_n)R(z). \tag{3.51}$$

Thus by an elementary comparison of the growth of the left-hand and the right-hand side of (3.51), we can deduce $R(z)\equiv\frac{1}{d^n}$,

$$b_1=1-\frac{1}{d^n}, \tag{3.52}$$

and

$$b_0=b_j=0 \quad (2\leqslant j\leqslant n+1). \tag{3.53}$$

From (3.39), (3.52) and (3.53) we can get

$$f(z)=\frac{\gamma_2 e^{dz}}{d^n}+\left(1-\frac{1}{d^n}\right)z. \tag{3.54}$$

On the other hand, from (3.38) and (3.53) we can get

$$f^{(n)}(z)=\gamma_2 e^{dz} \tag{3.55}$$

and

$$f^{(n+1)}(z)=\gamma_2 de^{dz}. \tag{3.56}$$

Substituting (3.55) and (3.56) into (3.33) we can get

$$\frac{\gamma_2 de^{dz}-z}{\gamma_2 e^{dz}-z}=d,$$

from which we can deduce $d=1$. From this and (3.54) we can get the conclusion (ii) of Theorem 1.2.

Theorem 1.2 is thus completely proved. □

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